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BUCKLING OF RECTANGULAR ISOTROPIC OR ORTHOGONAL ANISOTROPIC PLATES UNDER SHEAR STRESSES

E. Seydel

Translation of "Uber das Ausbeulen von rechteckigen, isotropen oder orthogonalanisotropen Platten bei Schubbeanspruchung," Ingenieur Archiv, Vol. 4, No. 2, 1933, pp. 169-191.

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16. Abstract

Equations are given for the critical shear stress t_{cr} on an isotropic rectangular plate and the resulting buckled surface. These equations are solved numerically by using finite trigonometric series as approximations. The results are described and diagrammed, and the closeness of the approximations is analyzed. Finally, the method is extended to the case of the orthogonally anisotropic plate.

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BUCKLING OF RECTANGULAR ISOTROPIC OR ORTHOGONAL ANISOTROPIC PLATES UNDER SHEAR STRESSES

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1. Delineation of the problem. Thin plates (sheet metal or plywood) are used as crosspieces for solid wall arch supports or, in aircraft construction, especially as the sheathing of wings or fuselage. If these construction parts are loaded, the originally flat plates, under a certain load, buckle. In most practical cases, the loads on the plates are generally shear stresses. /169*

As a contribution to the solution of the resulting problems, the following will be assumed: a completely flat, rectangular, thin, homogeneous plate of constant thickness is mounted at the edges without stress. The edge supports are rigid, and the camps holding the plate to the edge supports are made so that the plate cannot buckle out of its plane along the four straight edges. Along the four edges an evenly distributed shear force t, constant throughout, is applied and, if the force is small enough so that the plate remains flat, a shear stress which is constant throughout the entire surface of the plate is produced (see Fig. 1). The problem is now to determine the critical load t_{cr}, at which, under pure shear stress, the plate is no longer in stable equilibrium, that is, at which the plate no longer remains flat, but begins to buckle.[1] This problem is of importance not only for isotropic plates, but also for orthogonal-

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[1] Often this critical load is designated as the cracking load. But since the plate can remain capable of bearing a load after surpassing this critical load, we use the expression "buckle" instead of "crack" to indicate that, in practice, reaching of the critical load does not lead immediately to breaking of the plate, in constrast with a road under compression.

*Numbers in the margin indicate pagination in the foreign text.

anisotropic (orthotropic) plates. (In practice, such orthotropic plates are plywood sheets, and, with a certain approximation, also reinforced plates with suitably placed, relatively closely space supports, as well as corrugated steel plates. Sheet metal plates as well, because of the rolling process, show a certain orthogonal anisotropy, the effect of which, however, is so small that sheet metal slabs may be viewed as isotropic plates for technical purposes).

The problem just described has already been treated several times: Reissner [1, 2] and Timoshenko [3] have developed, in various ways, an approximate solution to the problem, starting with just isotropic plates. For a specific case of the problem, Southwell and Skan [4] gave an exact solution, namely, for the extreme case of an infinitely long isotropic plate, which was discussed by Bergmann and Reissner [2], and also in a supple- /170
mentary study [1], also for an orthotropic plate. Sekeriy-Tsenkovich, in connection with the work of Timoshenko, calculated the critical shear stress for a rectangular plywood plate (with given stiffness values), i.e. an example of an orthotropic plate. Sekeriy-Tsenkowich did what Timoshenko had done and used, following the procedure used by Bryan-Ritz in the investigation of the extreme case of an infinitely long plate, an expression different from that for the case of a rectangular plate of finite length, an expression which satisfied neither the differential equation, nor the boundary conditions. Bergmann and Reissner [5], as has been mentioned, took an approach different from Timoshenko's in deriving the calculation procedures, and, moreover, have taken the approximation calculation further.

This work, based on a suggestion of Professor Reissner, continues the work of Bergmann and Reissner. The work contains the following individual investigations:

The Reissner method is applied to orthotropic plates. The investigations of isotropic plates mentioned in the publications given above will be extended. For infinitely long plates, approximate values will be calculated from the same expression used in the treatment of plates of finite length. This approximate solution will be compared with the exact (Southwell-Skan) solution with regard to both the value of the critical load, and the shape of the buckled surface. In this case the precision of the method can be satisfactorily assessed. In another extreme case of rectangular plates, namely for the square plate, previous studies will be extended by pushing the approximation further; this is made possible first by the fact that the numerical treatment of this special case can be simplified for reasons of symmetry, and, second, by the fact that a procedure will be given by which the critical load and the coefficients for the series expansion of the function representing the buckled surface may be calculated, using successive approximations. It will turn out that these coefficients decrease rather rapidly, so the series appears to converge well. This is a circumstance which is particularly useful in deciding whether the expression given by Reissner and Timoshenko yields the correct result. In order to get a convenient numerical treatment for a plate with an arbitrary ratio between the sides, the number of terms in the series expansion of the buckling function should not be too large; for each ratio between the sides of the plate, terms with specific coefficients are found which provide the best solutions, i.e. those with the smallest buckling load. Finally, based on the calculations executed in this manner, a curve is given for the critical load as a function of the side ratio of the plate, which ought to agree with the

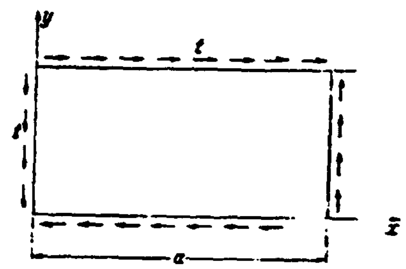

Fig. 1. Rectangular plate under evenly distributed shear force t applied along edges.

actual curve with an accuracy sufficient in practice. Similarly, corresponding calculations are carried out for the orthotropic plate, and the results of these calculations depicted graphically, so that this representation can be used to at least estimate the critical loads for orthotropic plates occurring in practice, and this immediately without computation. Simultaneously, the method by which the rather necessary more precise calculation for an orthotropic plate may be made will be indicated.

2. Trial solution for orthotropic plate. The differen- /171
tial equation of the orthotropic plate problem discussed here is [6, Eq. (10)]:

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} + 2t \frac{\partial^2 w}{\partial x \partial y} = 0. \tag{1}$$

x and y are the coordinates along two orthogonal axes coinciding with two sides of the rectangular plates; the other two sides have the coordinates x = a and y = b (cf. Fig. 1). D_1, D_2 and D_3 are the following abbreviations:

$$D_1 = \frac{(EJ)_x}{1-\nu_x\nu_y}, \qquad D_2 = \frac{(EJ)_y}{1-\nu_x\nu_y}, \qquad 2D_3 = \nu_x \frac{(EJ)_y}{1-\nu_x\nu_y} + \nu_y \frac{(EJ)_x}{1-\nu_x\nu_y} + 4(GJ)_{xy}. \tag{1a}$$

In these expressions $(EJ)_x$, $(EJ)_y$ and $4(GJ)_{xy}$ are the plate rigidities per unit length, namely

$(EJ)_x$: bending rigidity in x-direction (bending about the y-axis),
$(EJ)_y$: bending rigidity in y-direction (bending about the x-axis),
$4(GJ)_{xy}$: torsion rigidity[2]
ν_x and ν_y are the Poisson transverse strain coefficients

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[2] For a homogeneous, orthotropic plate of constant thickness δ (which will be the case for plywood boards, to a certain approximation),

$$(EJ)_x = E_x \frac{\delta^3}{12}; \qquad (EJ)_y = E_y \frac{\delta^3}{12}; \qquad 4(GJ)_{xy} = G \frac{\delta^3}{3}.$$

The smallest of the eigenvalues t corresponding to the various possible solutions of the differential equation, i.e., the eigenfunctions w, with a specific side ratio a/b and the prescribed boundary conditions (stressless support of the plate and no curvature along the edges) is the desired critical shear load t_{cr}.

The equation is solved with the aid of the Navier expression introduced by Timoshenko and Reissner[3]

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b}. \tag{2}$$

Each term in this series satisfies the prescribed boundary conditions. The coefficients A_{mn} can be determined not only by the Ritz method, but also (according to Reissner) in such fashion that the series formally satisfies the differential equation (1), and in fact, both methods yield the same determining equations. In order to obtain an equation for A_{mn}, the above expression for w in Eq. (2) is substituted in the differential equation, replacing m and n by r and s, and then the resulting equation is multiplied by sin $(m\,\pi\,x/a)\cdot\sin(n\,\pi\,y/b)$ and integrated from 0 to a and from 0 to b. Most of the integrals vanish, except for

$$\int_0^a \sin^2 m\pi \frac{x}{a}\, dx = \frac{a}{2},$$

$$\int_0^a \sin m\pi \frac{x}{a} \cos r\pi \frac{x}{a}\, dx = \frac{2a}{\pi} \frac{m}{m^2 - r^2},$$

if m $\pm$ r is odd, and except for the corresponding integrals in

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[3]Cf. Reference [2] on p. 1; in this work of Bergmann and Reissner, the derivation of the equations determining the coefficients A_{mn} and the assumptions under which this method is employed are described in more detail; therefore, in this work, the derivation will be just sketched.

the y-direction. A simple reformulation yields the following equation for A_{mn}: /172

$$A_{mn} + \frac{32}{\pi^4}\frac{t}{ab}\frac{mn}{D_1\frac{m^4}{a^4} + 2D_3\frac{m^2}{a^2}\frac{n^2}{b^2} + D_2\frac{n^4}{b^4}}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} A_{rs}\frac{r}{m^2-r^2}\frac{s}{n^2-s^2} = 0.$$

In order to obtain the critical shear load in dimensionless form [6], the following abbreviations are defined:

$$\vartheta = \frac{\sqrt{D_1 D_2}}{D_3} \quad \text{coefficient of the orthotropic plate} \qquad (3)$$

$$\beta_a = \frac{b}{a}\sqrt[4]{\frac{D_1}{D_2}}, \qquad c_a = \frac{t\left(\frac{b}{2}\right)^2}{\sqrt[4]{D_1 D_2^3}}, \qquad \varphi_a(m,n) = (m\beta_a)^4 + 2\frac{(m\beta_a)^2 n^2}{\vartheta} + n^4. \qquad (4a)$$

The governing equation for the coefficients A_{mn} is then:

$$A_{mn}\frac{\varphi_a(m,n)}{mn\beta_a} + \frac{128}{\pi^4}c_a\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} A_{rs}\frac{r}{m^2-r^2}\frac{s}{n^2-s^2} = 0. \qquad (5a)$$

This form of the equation is convenient for numerical evaluation when the coefficient $\theta \geq 1$, while if $\theta \leq 1$, is more convenient to set

$$\beta_b = \frac{b}{a}\sqrt{\frac{D_3}{D_2}}, \qquad c_b = \frac{t\left(\frac{b}{2}\right)^2}{\sqrt{D_2 D_3}}, \qquad \varphi_b(m,n) = \vartheta^2(m\beta_b)^4 + 2(m\beta_b)^2 n^2 + n^4 \qquad (4b)$$

and use the equations

$$A_{mn}\frac{\varphi_b(m,n)}{mn\beta_b} + \frac{128}{\pi^4}c_b\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} A_{rs}\frac{r}{m^2-r^2}\frac{s}{n^2-s^2} = 0. \qquad (5b)$$

In these equatic s (reglardless of whether the form (5a) or (5b) is used), the terms in the sum with even sums (or differences) m $\pm$ r or n $\pm$ s vanish. Therefore, as Bergmann and Reissner first showed in general, two independent systems of equations are obtained: one (I) containing the unknowns A_{mn} when the sum $(m + n)$ of the indices is even, and a second (II) when the sum of the indices $(m + n)$ is odd. In both cases the number of unknowns A_{mn} and of course the number of governing equations is infinitely large. Since the equations are homogeneous, the denominator determinant of the systems of equations must vanish; this is the condition from which c_a or c_b, i.e. the desired critical shear load, must be calculated. It is found that these infinite determinants do not have the properties previously required by mathematicians for a convergent calculation of unknowns. In particular, it is noteworthy that the otherwise usable iteration methods [8] seem to fail for Eqs. (5a) and (5b). Moreover, a practical, sufficiently accurate approximation cannot be made unless the unknowns A_{mn} of a system of equations can be divided into two groups in such a fashion that the first influential group contains only as many unknowns A_{mn} as can be determined conveniently when the unknowns A_{mn} of the second group are neglected. The problem is then to discover the block of indices (m, n) which is most influential. This block does not always have to stand at the beginning of the double series. The unknowns A_{mn} of the second group must then be so small that their influence on the critical shear load and on the unknowns of the first group will be negligibly small. If these prerequisites are satisfied, an approximation /173 is possible; however, for each new side ratio a/b and each new rigidity ratio θ (3), the unknowns A_{mn} in the first group involved in calculating the critical shear load must be determined all over again.

3. The infinitely long isotropic plate. In the equations developed in the previous section, the isotropic plate was a special case. Namely, for an isotropic, homogeneous plate of constant thickness δ,

$$D_1 = D_2 = D_3 = D = \frac{Et^3}{12(1-\nu^2)}. \tag{6}$$

where E is the modulus of elasticity of the material and ν is the transverse strain coefficient (Poisson's ratio). According to (3), $\theta = 1$, and in accordance with (4a) or (4b)

$$\left.\begin{aligned} \beta &= \frac{b}{a} \qquad (= \beta_a = \beta_b), \\ c_a &= \frac{t\left(\frac{b}{2}\right)^2}{D} \qquad (= c_b), \\ \varphi(m,n) &= [(m\beta)^2 + n^2]^2 (= \varphi_a(m,n) = \varphi_b(m,n)). \end{aligned}\right| \tag{7}$$

Temporarily, let us use the abbreviation

$$C = \frac{128}{\pi^4} c_a.$$

In that case (5a) or (5b) is replaced by the following equation for the isotropic plate

$$A_{mn} \frac{\varphi(m,n)}{mn\beta} + C \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} A_{rs} \frac{r}{m^2 - r^2} \frac{s}{n^2 - s^2} = 0. \tag{8}$$

If we now transpose to the limiting case $a \to \infty$, i.e. $\beta \to 0$ (infinitely long plate), Eq. (8) would supply the value $C = \infty$, i.e. an infinitely large critical shear load, as long as finite whole numbers were substituted for m and n (and correspondingly for r and s) and as long as the coefficients A_{mn} (A_{rs}) assumed finite, nonvanishing values. This is because more and more terms of higher order in m must be taken as the plate grows in the x-direction; namely, the assumption of finite m (r) with an infinitely long plate means an infinitely large half-wavelength of the buckled surface. Therefore, in order to obtain a finite half-wavelength for the buckled surface, it seems logical to assume values m (r) of the same order of magnitude as the length of the plate, i.e. as the same order of magnitude as $1/\beta$. If $1/\beta$ goes to infinity, the values m and r must also be taken to be of the order of magnitude infinity. We presume that it is sufficient to examine a limited number of coefficients A_{mn}. In order to express the fact that m and r are in general different values of the same order of magnitude, we substitute in (8)

$$m = \frac{\alpha}{\beta} + m_1, \qquad r = \frac{\alpha}{\beta} + r_1,$$

where m_1 and r_1 are integers (or equal to 0), while α is a finite number (α can only assume values such that α/β is an integer). Passing to the limit $\beta \to 0$ and using the abbreviation

$$\alpha/\beta = p,$$

we obtain the equation

$$A_{(p+m_1)n} \frac{\varphi(\alpha, n)}{\alpha n} + C \frac{1}{\alpha} \sum_{r_1=-\infty}^{+\infty} \sum_{s=1}^{\infty} A_{(p+r_1)s} \frac{1}{m_1 - r_1} \frac{s}{n^2 - s^2} = 0, \tag{9}$$

where /174

$$\varphi(x, n) = (x^2 + n^2)^2$$

The system of equations given by Eq. (9) also breaks down into two systems of equations: one for even sums $(m_1 + n)$ and a second for odd sums $(m_1 + n)$. The meaning of α is demonstrated by the following consideration: the function $\sin(m \pi x/a)$ is a sinusoidal (wave) line with the half-wavelength

$$l = \frac{a}{m} \tag{10}$$

Using the above expression for m and $a = b/\beta$, this can also be written

$$l = \frac{b}{\alpha + m_1 \beta}.$$

For $\beta \to 0$,

$$\frac{l}{b} = \frac{1}{\alpha}. \tag{11}$$

The reciprocal of α is therefore the ratio of the half-wavelength to the width of the infinitely long plate. While α can only assume specific values (since m must always be an integer) if the plate length a is finite, the variable α can assume any arbitrary positive value in a limiting case $a \to \infty$. This is also consistent with the results of the Southwell-Skan exact solution of the differential equation.

The coefficients C of the shear forces (eigenvalues) t, which are possible solutions of the differential equation, are roots of an equation obtained by setting the denominator determinant of the system (9) equal to 0. In practice, the calculation can be made only for a finite number of equations with a finite number of unknowns A. The unknowns A of each system of equations (with odd or even $(m_1 + n)$) can be broken down into two groups: one group with even index n and a second group with odd n. (Analogously, the groups could be distinguished on the basis of m_1). In each equation of the two systems, there is only one unknown A of one of the groups, while all the other unknowns A are of the other group. Accordingly, there are two groups of equations in each system. Since C^{-1} does not occur as a factor in one group, $C \cdot A$ (instead of A) is introduced as an unknown in one group. The system of coefficients of the system of equations then has the form depicted in Table 1, which illustrates the system with odd $(m_1 + n)$. Only the equations for the indices (p-3) through (p+3) (in the longitudinal direction) and n = 1 through n = 3 (in the transverse direction) are written. The coefficients with larger m_1 or n are assumed to be negligible. The error in C or c_a produced by this elimination should be compared with the exact value of C or c_a known in this case. In the system with odd $(m_1 + n)$ we now have

$$A_{(p + m_1)n} = + A_{(p - m_1)n}, \text{ when n is odd,}$$

$$A_{(p + m_1)n} = - A_{(p - m_1)n}, \text{ when n is even.}$$

Taking this into accoun. and expressing the unknowns $A_{(p+2)1}$, $A_{(p+2)3}$, A_{p1} and A_{p3} in terms of $A_{(p+1)2}$ and $A_{(p+3)2}$ (first group of equations), and substituting this in the second group of equations, we finally obtain just two equations with the unknowns $A_{(p+1)2}$ and $A_{(p+3)2}$, i.e. just a second-order determinant, and, by setting it equal to 0, a quadratic equation for $x = C^{-2}$. A determinant with just two terms is also obtained when even more unknowns A are aken into consideration, namely with the indices (p - 4) through (p + 4) and n = 1 through 3 or the indices (p - 2) through (p + 2) in the longitudinal direction, and n = 1 through 5 in the transverse direction (i.e. five indices in each direction). In the first case (i.e. with nine indices in the longitudinal direction and three indices in the transverse direction), we obtain

$$c_a = 0{,}161492 \frac{\alpha + \frac{4}{\alpha}}{\sqrt{\left[\frac{1}{9(\alpha^2+1)}\right]^2 + \left[\frac{1}{5(\alpha^2+9)}\right]^2}}. \qquad (12)$$

This function c_a has a minimum at $\alpha = 0.7995$. In the second case (five indices in each direction), the formula for c_a is not as simple. The first case yields more accurate values, and cannot be distinguished from the true curve in the graph in Fig. 2. This diagram also shows the curve obtained with three indices each in the longitudinal and transverse directions. For the precise curve (corresponding to the Southwall-Skan solution), the minimum $c_a = 13.165$ is between $\ell/b = 1.235$ and 1.257 [6, Table I] near $\ell/b = 1.25$ (i.e. $\alpha = 0.8$). The value given above, $\alpha = 0.7995$, for the approximate curve is in excellent agreement with this figure. Apart from the approximation already discussed, further approximations with smaller values of m_1 and n have been made for $\alpha = 0.8$ and the results collected in Table 2. For the longitudinal direction, only the number of (successive) indices used in the approximation is given, since the result is independent of whether the indices are taken from (p + 1) through (p + 9) or from (p - 4) through

TABLE 1. SYSTEM OF COEFFICIENTS FOR EQ. (9). (INFINITELY LONG PLATE)

$$\frac{2(\alpha^2+n^2)^2}{\alpha n} A_{(p+m_1)n} + C \sum A_{(p+r_1)s} \frac{1}{m_1-r_1} \frac{s}{n^2-s^2} = 0$$

$(p+m_1)n$ \ $A_{(p+r_1)s}$	$A_{(p-2)1}$	$A_{(p+2)1}$	$A_{(p-2)3}$	$A_{(p+2)3}$	A_{p1}	A_{p3}	$C\cdot A_{(p-1)2}$	$C\cdot A_{(p+1)2}$	$C\cdot A_{(p-3)2}$	$C\cdot A_{(p+3)2}$
$(p-2)\cdot 1$	$\frac{2(\alpha^2+1)^2}{\alpha\cdot 1}$	0	0	0	0	0	$\frac{1}{-1}\cdot\frac{2}{-3}$	$\frac{1}{-3}\cdot\frac{2}{-3}$	$\frac{1}{+1}\cdot\frac{2}{-3}$	$\frac{1}{-5}\cdot\frac{2}{-3}$
$(p+2)\cdot 1$	0	$\frac{2(\alpha^2+1)^2}{\alpha\cdot 1}$	0	0	0	0	$\frac{1}{+3}\cdot\frac{2}{-3}$	$\frac{1}{+1}\cdot\frac{2}{-3}$	$\frac{1}{+5}\cdot\frac{2}{-3}$	$\frac{1}{-1}\cdot\frac{2}{-3}$
$(p-2)\cdot 3$	0	0	$\frac{2(\alpha^2+9)^2}{\alpha\cdot 3}$	0	0	0	$\frac{1}{-1}\cdot\frac{2}{+5}$	$\frac{1}{-3}\cdot\frac{2}{+5}$	$\frac{1}{+1}\cdot\frac{2}{+5}$	$\frac{1}{-5}\cdot\frac{2}{+5}$
$(p+2)\cdot 3$	0	0	0	$\frac{2(\alpha^2+9)^2}{\alpha\cdot 3}$	0	0	$\frac{1}{+3}\cdot\frac{2}{+5}$	$\frac{1}{+1}\cdot\frac{2}{+5}$	$\frac{1}{+5}\cdot\frac{2}{+5}$	$\frac{1}{-1}\cdot\frac{2}{+5}$
$p\cdot 1$	0	0	0	0	$\frac{2(\alpha^2+1)^2}{\alpha\cdot 1}$	0	$\frac{1}{+1}\cdot\frac{2}{-3}$	$\frac{1}{-1}\cdot\frac{2}{-3}$	$\frac{1}{+3}\cdot\frac{2}{-3}$	$\frac{1}{-3}\cdot\frac{2}{-3}$
$p\cdot 3$	0	0	0	0	0	$\frac{2(\alpha^2+9)^2}{\alpha\cdot 3}$	$\frac{1}{+1}\cdot\frac{2}{+5}$	$\frac{1}{-1}\cdot\frac{2}{+5}$	$\frac{1}{+3}\cdot\frac{2}{+5}$	$\frac{1}{-3}\cdot\frac{2}{+5}$
$(p-1)\cdot 2$	$\frac{1}{+1}\cdot\frac{1}{+3}$	$\frac{1}{-3}\cdot\frac{1}{+3}$	$\frac{1}{+1}\cdot\frac{3}{-5}$	$\frac{1}{-3}\cdot\frac{3}{-5}$	$\frac{1}{-1}\cdot\frac{1}{+3}$	$\frac{1}{-1}\cdot\frac{3}{-5}$	$\frac{2(\alpha^2+4)^2}{\alpha\cdot 2}C^{-1}$	0	0	0
$(p+1)\cdot 2$	$\frac{1}{+3}\cdot\frac{1}{+3}$	$\frac{1}{-1}\cdot\frac{1}{+3}$	$\frac{1}{+3}\cdot\frac{3}{-5}$	$\frac{1}{-1}\cdot\frac{3}{-5}$	$\frac{1}{+1}\cdot\frac{1}{+3}$	$\frac{1}{+1}\cdot\frac{3}{-5}$	0	$\frac{2(\alpha^2+4)^2}{\alpha\cdot 2}C^{-1}$	0	0
$(p-3)\cdot 2$	$\frac{1}{-1}\cdot\frac{1}{+3}$	$\frac{1}{-5}\cdot\frac{1}{+3}$	$\frac{1}{-1}\cdot\frac{3}{-5}$	$\frac{1}{-5}\cdot\frac{3}{-5}$	$\frac{1}{-3}\cdot\frac{1}{+3}$	$\frac{1}{-3}\cdot\frac{3}{-5}$	0	0	$\frac{2(\alpha^2+4)^2}{\alpha\cdot 2}C^{-1}$	0
$(p+3)\cdot 2$	$\frac{1}{+5}\cdot\frac{1}{+3}$	$\frac{1}{+1}\cdot\frac{1}{+3}$	$\frac{1}{+5}\cdot\frac{3}{-5}$	$\frac{1}{+1}\cdot\frac{3}{-5}$	$\frac{1}{+3}\cdot\frac{1}{+3}$	$\frac{1}{+3}\cdot\frac{3}{-5}$	0	0	0	$\frac{2(\alpha^2+4)^2}{\alpha\cdot 2}C^{-1}$

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(p + 4); in a transverse direction, the first index is always n = 1. Table 2 shows that in order to obtain a good approximation, it is best to take more terms in the longitudinal direction than in the transverse direction, and that the error of the best approximation (nine longitudinal indices, three transverse indices) is less than 0.5%. For larger values of α, the errors in the approximation with this expression are substantially larger, e.g. 4.65% for $\alpha = 3$ and 13% for $\alpha = 4$, while the approximation with five longitudinal and transverse indices yields an error of only 0.65% even for $\alpha = 4$. The smaller the wavelength (i.e. the larger α), the greater the influence of the number of indices n in the transverse direction.

In the system of equations with even sum $(m_1 + n)$, the same value c_a is obtained as in the corresponding system of equations with odd sum $(m_1 + n)$ with a given number of indices in the longitudinal and transverse directions. The buckled surface represented by the function w, the calculation of which is yet to come, is the same in both cases; there is only a difference in the position of the origin of the /177

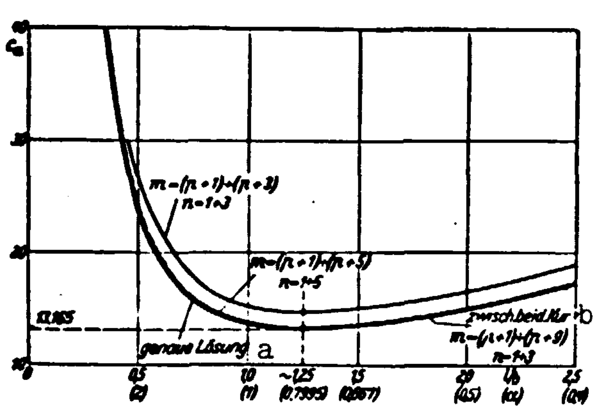

Fig. 2. Coefficient c_a of the critical shear load of an infinitely long, isotropic plate as a function of ℓ/b (ℓ = half-wavelength of the buckled surface, b = width of plate). Comparison of various approximate solutions with exact solution. (The best of the approximate solutions virtually coincides with the exact solution.) The number p is infinitely large, corresponding to the number of waves.

Key: a. Exact solution
b. Between two curves

TABLE 2. COEFFICIENT $c_a = t_{cr}(b/2)^2/D$ OF CRITICAL SHEAR LOAD IN VARIOUS APPROXIMATIONS FOR INFINITELY LONG PLATE AS COMPARED WITH EXACT VALUE $c_a = 13.165$

Number of indices of A_{mn} taken in		c_a approximated	Deviation from $c_a = 13.165$ in %
Longitudinal direction	Transverse direction		
3	3	14.678	11.65
4	2	14.145	7.5
5	2	13.9	5.6
5	3	13.281	0.88
5	5	13.235	0.53
9	3	13.219	0.41

coordinate system, lying in one case on a nodal line, and in the other case halfway between two nodal lines. A corresponding difference also turns up in the Southwall-Skan exact solution, and in fact between the real and imaginary parts of the function w.

In order to depict the form of the buckled surface, the function w must be calculated numerically by Eq. (2); however, this cannot be done directly for an infinitely long plate, since the origin of the coordinate system is at infinity, so that at first, no specific value can be given for sin $(m \pi x/a)$. The formula must be developed further. For this purpose, the function $w_{am} = \sin (m \pi x/a)$ for a relatively long plate of length a will be considered, m being so large that the half-wavelength of this function is on the order of the plate width. In the graphic representation of w_{am} in Fig. 3, the middle half-wave is emphasized (an odd number is assumed for m.) The functions $w_{a(m+1)}$ $w_{a(m+2)}$ and $w_{a(m+3)}$ with 1, 2 and 3 higher half-wave numbers than w_{am} are depicted. The larger the number m of the half-waves lying within the length a, the smaller the differences in half-wavelengths in these four cases; in the limiting case $m \to \infty$, in which case $\alpha \to \infty$ at the same time, the half-wavelength must be equal in all four cases (cf. Eq. (11) unless the half-wavelength vanishes, and, if the origin is taken at the beginning of one of the center half-waves, the four functions depicted in Fig. 3 can be written:

$$1.\ \sin\pi\frac{x}{l};\quad 2.\ \cos\pi\frac{x}{l};\quad 3.\ -\sin\pi\frac{x}{l};\quad 4.\ -\cos\pi\frac{x}{l}.$$

by transposing to the limiting case $\alpha \to \infty$ and keeping Eq.(10) in mind. To calculate the buckling function -- cf. Eq. (2) -- for the infinitely long plate, we obtain the equation:

$$w = \sin\pi\frac{x}{l}\left[\sum_{\substack{\pm i=0,1,2,3\ldots\\ n=1,2,3\ldots}} A_{(p+4i)n}\sin n\pi\frac{y}{b} - \sum_{\substack{\pm i=0,1,2,3\ldots\\ n=1,2,\ldots}} A_{(p+2+4i)n}\sin n\pi\frac{y}{b}\right]$$
$$+\cos\pi\frac{x}{l}\left[\sum_{\substack{\pm i=0,1,2,3\ldots\\ n=1,2,3\ldots}} A_{(p+1+4i)n}\sin n\pi\frac{y}{b} - \sum_{\substack{\pm i=0,1,2,3\ldots\\ n=1,2,3\ldots}} A_{(p+3+4i)n}\sin n\pi\frac{y}{b}\right].$$

Here, ℓ is obtained from (11). The number p, which occurs only in the indices of the coefficient A, and which was introduced in the derivation of Eq. (9), is infinitely large, in accordance with the number of waves.

Using this equation, the buckled shape was calculated under the assumption $\alpha = 0.8$ ($\ell = 1.25$ b) for five indices in the longitudinal direction (p - 2 through p + 2) and three indices in the transverse direction (n = 1 through 3), in order to demonstrate the difference in the buckled shape obtained by the exact method (Southwall-Skan) and by the approximation. This difference is shown in the contours depicted in Fig. 4; the solid lines correspond to the exact solution. In the left half of the diagram, the broken lines represent the contours /178

TABLE 3. CO[illegible]FICIENTS A_{mn} OF A BUCKLED SURFACE AND COEFFICIENT c_a OF BUCKLING [illegible]OAD ACCORDING TO VARIOUS APPROXIMATIONS FOR THE INFINITELY LONG PLATE

Number of indices in the Longitudinal Direction	Number of indices in the Transverse Direction	A_{p1}	$A_{(p\pm2)1}$	$A_{(p\pm1)2}$	$A_{(p\pm3)2}$	$A_{(p\pm4)1}$	A_{p3}	$A_{(p\pm2)3}$	$A_{(p\pm1)4}$	A_{p5}	$A_{(p\pm4)3}$	$A_{(p\pm2)5}$	$c_a = \frac{t_{cr}(b/2)^2}{D}$
5 ($m = p-2$ to $p+2$)	3 ($n = 1-3$)	+ 0.5703	− 0,1901	∓ 0,1648	—	—	− 0,0297	+ 0,0099	—	—	—	—	13,28
5 ($m = p-2$ to $p+2$)	5 ($n = 1-5$)	+ 0.5732	− 0,1911	∓ 0,1649	—	—	− 0,0288	+ 0,0096	∓ 0,0033	− 0,0018	—	+ 0,0006	13,235
9 ($m = p-4$ to $p+4$)	3 ($n = 1-3$)	+ 0.4326	− 0.2356	∓ 0.1380	∓ 0.0373	+ 0.0234	− 0,0225	+ 0,0123	—	—	− 0,0012	—	13,29

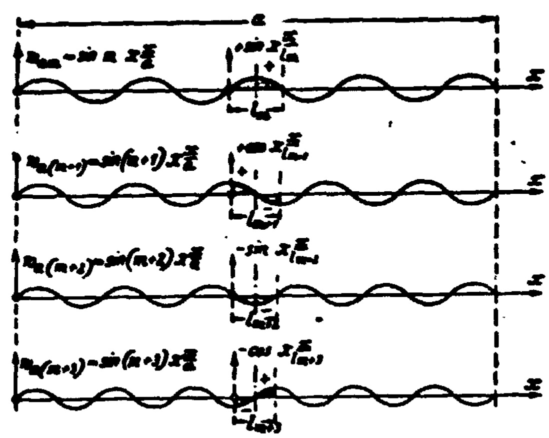

Fig. 3. Sinusoidal lines with various numbers of periods in a specific length a. (Passing to the limit $a \to \infty$ and $m \to \infty$, we obtain $\ell_m = \ell_{m+1} = \ell_{m+2} = \ell_{m+3}$).

corresponding to the approximation described above, whenever they differ from the exact solution. The differences are greatest for the nodal line ($w = 0$). In the left half of the diagram, the nodal line resulting from a further approximation (five indices in each direction) is plotted as a dotted line, closer to the exact line than the broken line. The coefficients A_{mn} (or $A_{(p \pm m_1)n}$) with which the buckled surface was calculated are collected in Table 3, which also contains the values A_{mn} for the approximation not used in the graphic representation in nine indices in the longitudinal direction and three indices in the transverse direction. The coefficients A_{mn} can be determined from system (9) up to a factor identical in all coefficients, which was chosen in all calculations so that the maximum buckling was $w_{max} = 1.0$. The values $A_{(m \pm m_1)n}$ decreased as the indices m_1 and n increase; it can also be recognized from the table that it is better to take a larger number of indices in the longitudinal direction than in the transverse direction. A conspicuous feature is the influence of the inclusion of the term $A_{(p \pm 3)2}$, which as opposed to the first two approximations given in Table 3, causes a change in A_{p1} and $A_{(p \pm 2)1}$ which is greater than $A_{(p + 3)2}$ itself.

Comparisons of both the buckled surfaces, obtained by the approximation and by the exact solution for the infinitely long

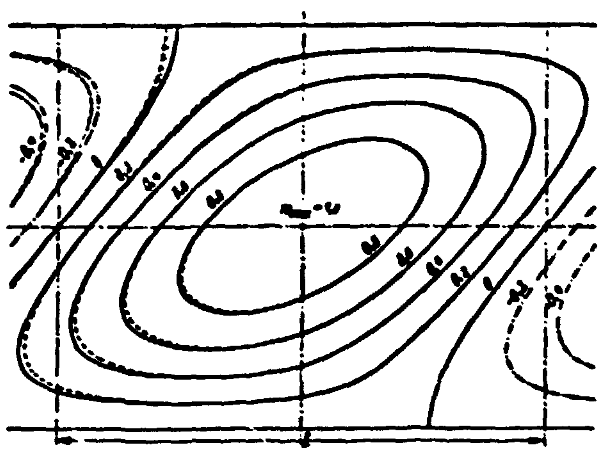

Fig. 4. Contours of buckled surface of infinitely long isoptropic plate. Comparison of approximations with exact solution for $\ell/b = 1.25$.

-·-·-·-·-·-· Exact solution

------ Approximate solution: (p + 1) through (p + 5); n = 1 through 3 (only on the left half of the diagram).

...... Approximate solution: (p + 1) through (p + 5); n = 1 through 5 (only for a nodal line w = 0).

plate, and of the corresponding critical shear loads show excellent agreement, which can be considered sufficiently precise for practical purposes. Although the approximation does not go beyond cases yielding a complete determinate of two terms for the determination of the critical shear load, the error in this approximation is less than 0.5%; restricting ourselves to working with a one-term determinant (p - 2 through p + 2; n = 1 through 3) still does not make the errors greater than 1%. These results were obtained with rather little computation, making use of certain simplifications valid for the special case of the infinitely long plate. In the other limiting case of the problem, namely the square plate, it will also be possible to make certain simplifications.

4. The square plate. In the case of a square plate, the same indices are chosen in the x- and y-directions because of symmetry; likewise because of symmetry, $A_{mn} = A_{nm}$. This relationship simplifies the calculation. These studies of Bergmann and Reissner [2] showed that the system of equations for the coefficients A_{mn} with even sum of indices (m + n) (Case I)

TABLE 4. SYSTEM OF COEFFICIENTS FOR EQS. (8). (SQUARE PLATE)

$$A_{mn}\frac{(m^2+n^2)^2}{mn} + C\sum A_{rs}\frac{r}{m^2-r^2}\,\frac{s}{n^2-s^2} = 0$$

A_{rs} / mn	A_{11}	A_{13}	A_{33}	A_{15}	A_{35}	A_{55}	$C\cdot A_{22}$	$C\cdot A_{24}$	$C\cdot A_{44}$
1·1	4	0	0	0	0	0	$\frac{2}{-3}\cdot\frac{2}{-3}$	$\left(\frac{2}{-3}\cdot\frac{4}{-15}+\frac{4}{-15}\cdot\frac{2}{-3}\right)$	$\frac{4}{-15}\cdot\frac{4}{-15}$
1·3	0	$\frac{100}{3}$	0	0	0	0	$\frac{2}{-3}\cdot\frac{2}{+5}$	$\left(\frac{2}{-3}\cdot\frac{4}{-7}+\frac{4}{-15}\cdot\frac{2}{+5}\right)$	$\frac{4}{-15}\cdot\frac{4}{-7}$
3·3	0	0	36	0	0	0	$\frac{2}{+5}\cdot\frac{2}{+5}$	$\left(\frac{2}{+5}\cdot\frac{4}{-7}+\frac{4}{-7}\cdot\frac{2}{+5}\right)$	$\frac{4}{-7}\cdot\frac{4}{-7}$
1·5	0	0	0	$\frac{676}{5}$	0	0	$\frac{2}{-3}\cdot\frac{2}{+21}$	$\left(\frac{2}{-3}\cdot\frac{4}{+9}+\frac{4}{-15}\cdot\frac{2}{+21}\right)$	$\frac{4}{-15}\cdot\frac{4}{+9}$
3·5	0	0	0	0	$\frac{1156}{15}$	0	$\frac{2}{+5}\cdot\frac{2}{+21}$	$\left(\frac{2}{+5}\cdot\frac{4}{+9}+\frac{4}{-7}\cdot\frac{2}{+21}\right)$	$\frac{4}{-7}\cdot\frac{4}{+9}$
5·5	0	0	0	0	0	100	$\frac{2}{+21}\cdot\frac{2}{+21}$	$\left(\frac{2}{+21}\cdot\frac{4}{+9}+\frac{4}{+9}\cdot\frac{2}{+21}\right)$	$\frac{4}{+9}\cdot\frac{4}{+9}$
2·2	$\frac{1}{+3}\cdot\frac{1}{+3}$	$\left(\frac{1}{+3}\cdot\frac{3}{-5}+\frac{3}{-5}\cdot\frac{1}{+3}\right)$	$\frac{3}{-5}\cdot\frac{3}{-5}$	$\left(\frac{1}{+3}\cdot\frac{5}{-21}+\frac{5}{-21}\cdot\frac{1}{+3}\right)$	$\left(\frac{3}{-5}\cdot\frac{5}{-21}+\frac{5}{-21}\cdot\frac{3}{-5}\right)$	$\frac{5}{-21}\cdot\frac{5}{-21}$	$16C^{-1}$	0	0
2·4	$\frac{1}{+3}\cdot\frac{1}{+15}$	$\left(\frac{1}{+3}\cdot\frac{3}{+7}+\frac{3}{-5}\cdot\frac{1}{+15}\right)$	$\frac{3}{-5}\cdot\frac{3}{+7}$	$\left(\frac{1}{+3}\cdot\frac{5}{-9}+\frac{5}{-21}\cdot\frac{1}{+15}\right)$	$\left(\frac{3}{-5}\cdot\frac{5}{-9}+\frac{5}{-21}\cdot\frac{3}{+7}\right)$	$\frac{5}{-21}\cdot\frac{5}{-9}$	0	$50C^{-1}$	0
4·4	$\frac{1}{+15}\cdot\frac{1}{+15}$	$\left(\frac{1}{+15}\cdot\frac{3}{+7}+\frac{3}{+7}\cdot\frac{1}{+15}\right)$	$\frac{3}{+7}\cdot\frac{3}{+7}$	$\left(\frac{1}{+15}\cdot\frac{5}{-9}+\frac{5}{-9}\cdot\frac{1}{+15}\right)$	$\left(\frac{3}{+7}\cdot\frac{5}{-9}+\frac{5}{-9}\cdot\frac{3}{+7}\right)$	$\frac{5}{-9}\cdot\frac{5}{-9}$	0	0	$64C^{-1}$

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yielded the smaller critical shear load (c_a). Therefore, with a square plate, we will deal just with this case. The coefficients calculated for the unknowns A_{mn} using (8) for $\beta = 1$ (i.e. $a = b$) are listed in Table 4, including all unknowns A_{mn} with indices 1 through 5. Once again, as with the system of equations for the infinitely long plate, the unknowns can be divided into two groups: those with odd indices m and n (A_{11}, A_{13}.....A_{55}) and those with even indices m and n (A_{22}, A_{24}, A_{44}). By substituting equations of the first group into the Eqs. (5a) or (5b) formulated for A_{22}, A_{24} and A_{44}, we finally obtained just three equations in these three unknowns. (Eqs. (5a) and (5b) are formulated in general form for A_{mn}.) C, i.e. the critical shear load, is then calculated by solving the following third-order determinant, set equal to zero:

$$\begin{vmatrix} \begin{matrix} 17366{,}596 \\ +16\cdot 10^{12}\cdot C^{-2} \end{matrix} & -2896{,}683 & -2\ .1{,}6146 \\ -724{,}17075 & \begin{matrix} -7047{,}572 \\ +50\cdot 10^{12}\cdot C^{-2} \end{matrix} & +1791{,}7703 \\ -680{,}40366 & +1791{,}7703 & \begin{matrix} 4961{,}1733 \\ +64\cdot 10^{12}\cdot C^{-2} \end{matrix} \end{vmatrix} = 0.$$

Expanding this determinant yields the equation: /181

$$51{,}200(10^3\cdot C^{-2})^3 - 66{,}755760(10^3\cdot C^{-2})^2 + 12{,}422262(10^3\cdot C^{-2}) - 0{,}52093462 = 0.$$

The three roots ($10^3 \cdot C^{-2}$) of this equation are real and positive; the largest of these roots has the value $10^3 \cdot C^{-2} = 1.089822$, so that $C = 30.292$, and $c_a = \pi^4 C/128 = 23.05$. The other two roots ($10^3 \cdot C^{-2}$) yield values of c_a which are about 2.7 and 4.2 times as large.

Once C has been determined, the unknowns A_{nm} can then be calculated. The result of this calculation is shown in Table

5, and again the coefficients A_{nm} have been calculated in such a fashion that the maximum buckling occurring in the center of the plate is $w_{max} = 1.0$. The table also gives the corresponding results for the cases in which the indices m and n run not from 1 through 5 but instead through 2, 3 or 4 and finally through 6. When the indices run only through 2 or 3, C is obtained from a simple determinant (for A_{22}) instead of a third-order determinant; if the indices run through 6, one would eventually have to solve a complete sixth-order determinant if one proceeded in the same manner as above. In order to avoid this, another method was employed (successive approximations of the unknowns A_{nm} and C), which, however, does not supply the precise (approximate) value C and the corresponding values A_{nm}, which would be obtained from the sixth-order determinant; nevertheless, with relatively little computation, this method will yield values sufficiently accurate to indicate that the results of the computation with indices 1-6 differs very little from the computation with indices 1-5. In particular, it is evident that the new unknown A_{26}, A_{46} and A_{66} are substantially smaller than the other unknowns in the computation and that there is only a very small decrease in the approximate c_a. This calculation was performed as follows: the values A_{mn} and the value C calculated for the indices 1 through 5 were substituted as first approximations in the new system of equations for the indices 1 through 6. This system involved the new unknowns A_{26}, A_{46}, and A_{66}, which all belong to the second group of unknowns (with even indices). For these new unknowns, the first approximations multiplied by C^{-1} ($C^{-1} A_{26}$, $C^{-1} A_{46}$ and $C^{-1} A_{66}$) were calculated from the new equations containing only one of these unknowns together with unknowns of the first group (with odd indices). Since the unknowns A_{mn} are determined only up to an arbitrary factor, one of the A_{mn} could be chosen arbitrarily, e.g. we set $A_{11} = 1$ and then introduced the unknowns $x_{mn} = A_{mn}/A_{11}$ to replace the A_{mn}. Once first approximations have been found for x_{26}, x_{46} and x_{66}, /182
a new (second) approximation C^{-2} ($= C^{-2} x_{11}$) can be calculated

TABLE 5. COEFFICIENT A_{nm} OF THE BUCKLED SURFACE AND COEFFICIENT c_a OF THE BUCKLING LOAD ACCORDING TO VARIOUS APPROXIMATIONS FOR THE SQUARE PLATE

m and n = 1 –	A_{11}	A_{22}	A_{13}	A_{33}	A_{24}	A_{44}	A_{15}	A_{35}	A_{55}	A_{26}	A_{46}	A_{66}	$c_a = \frac{t_{ke}(b/a)^2}{D}$
2	+ 1.0	− 0.25	—	—	—	—	—	—	—	—	—	—	47.40
3	+ 0.8446	− 0.2488	− 0.0608	+ 0.0338	—	—	—	—	—	—	—	—	43.25
4	+ 0.8478	− 0.2477	− 0.0594	+ 0.0335	− 0.0025	− 0.0031	—	—	—	—	—	—	43.20
5	+ 0.8613	− 0.2525	− 0.0599	+ 0.0333	− 0.0038	− 0.0026	− 0.0039	+ 0.0037	+ 0.0009	—	—	—	43.05
6	+ 0.8609	− 0.2522	− 0.0597	+ 0.0333	− 0.0037	− 0.0026	− 0.0038	+ 0.0034	+ 0.000[illegible]	− 0.0005	− 0.0008	− 0.0001	> 43.0[illegible]

from the first equation (the equation for A_{11}) and this second approximation will depend only on the values x_{mn} of the second group. Next, the unknowns of the second group multiplied by C^{-2} ($C^{-2}\,x_{13}$, $C^{-2}\,x_{33}$, $C^{-2}\,x_{15}$ etc) were calculated and, once the second approximation x_{mn} of the first group has been calculated (after division by the second approximation C^{-2}), the second approximations x_{mn} of the second group, multiplied by C^{-1}, ($C^{-1}\,x_{22}$, $C^{-1}\,x_{24}$ etc) were calculated. The procedure was repeated until the unknowns x_{mn} and C no longer changed appreciably. In order for the method to be usable, the new x_{mn} must be relatively small and the changes induced by their inclusion in the other values x_{mn} and in C must also be relatively small. The method was also employed in appropriate fashion in passing from the system of equations with indices 1-3 to the system with 1-4, and from the latter system to that with indices 1-5. Since in these two cases, exact solutions of the systems of equations have also been calculated, the results of these solutions were compared with the successive approximation method. The comparison showed that in the fourth approximation of the latter method, the results of both methods agreed to four places, and usually five. The results of four approximations for indices 1-6 are listed in Table 6.

Based on the approximation c_a to the critical shear load calculated for the square plate, it is reasonable to assume that the exact value c_a will not be much less than the value c_a = 23.05 obtained in the calculation with the indices 1 through 5. Moreover, including more indices does not seem to change the buckled surface itself very much, since the additional coefficients A_{mn} are relatively small and the approximations /183 to the buckled surface using indices 1 through 5 and 1 through 3 differ very little, as shown in Fig. 5. Hence it seems legitimate to suppose that including more indices in the approximation will not bring about any substantial changes in the buckled surface. Also, as a comparison of the coefficients

TABLE 6. CALCULATION OF x_{mn} (= A_{mn}/A_{11}) AND C, THE CRITICAL SHEAR FORCE ($C = 128\ t_{cr}\ (b/2)^2/\pi^4 D$) BY SUCCESSIVE APPROXIMATION FOR THE SQUARE PLATE, USING COEFFICIENTS A_{mn} WITH INDICES m = 1 - 6 AND n = 1 - 6.

	1. Approximation	2. Approximation	3. Approximation	4. Approximation
C^2	916.7743	917.9137	917.8480	917.8320
C	30.2783	30.29709	30.29606	30.29574
x_{13}	− 0.069348	− 0.069355	− 0.069358	− 0.06935[illegible]
x_{22}	+ 0.038636	+ 0.038676	+ 0.038682	+ 0.038682
x_{15}	− 0.004477	− 0.004473	− 0.004473	− 0.004472
x_{24}	+ 0.003971	+ 0.003962	+ 0.003962	+ 0.003962
x_{26}	+ 0.001039	+ 0.001042	+ 0.001042	+ 0.001042
x_{33}	− 0.293019	− 0.292865	− 0.292882	− 0.292885
x_{35}	− 0.004361	− 0.004308	− 0.004300	− 0.004209
x_{44}	− 0.002926	− 0.003001	− 0.003007	− 0.003
x_{46}	− 0.000543	− 0.000570	− 0.000569	− 0.000
x_{55}	− 0.000981	− 0.000970	− 0.000970	− 0.000
x_{66}	− 0.000306	− 0.000301	− 0.000301	− 0.0003

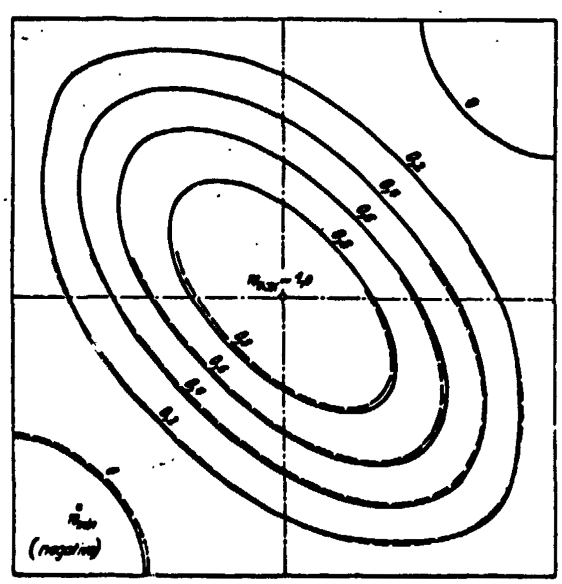

Fig. 5. Contours of buckled surface of isotropic square plate. Comparison of two approximations:
______ m = 1 - 5; n = 1 - 5.
\- - - - - - m = 1 - 3; n = 1 - 3 (only in the lower half of the diagram)

A_{mn} (Tables 3 and 5) indicates, the series for the buckling function appears to converge better for the square plate than for the infinitely long plate. In the latter case, taking five indices in both directions gave an error of only 0.5% in the critical shear load.

5. Rectangular plate of arbitrary proportions. a) Calculation of coefficient c_a of critical shear load. If in the formula

for calculating the shear load analogous to Eq. (7)

$$t_{cr} = c_a \ D \ (b/2)^{-2} \tag{13}$$

b is always taken to be the short side of the rectangular plate, it is only necessary to find the coefficient c_a for values $\beta = b/a$ which lie between 0 and 1. These two extreme cases for the range of β, representing the infinitely long plate and the square plate, have already been discussed in detail in the previous sections. The calculation for arbitrary data could now be carried out as it was done for the square plate. If four or five indices (1 - 4 or 1 - 5) each are taken in the longitudinal and transverse directions, one would eventually have to solve a complete fourth-order determinant (instead of the third-order determinant for $\beta = 1$) in Case I (since $A_{24} \neq A_{42}$). With indices 1 - 4, this calculation has been carried out by Bergmann and Reissner, and the results compared with the result for indices 1 through 3 in Fig. 3 of their work [2] (as well as in their Table 3). This comparison shows that the difference in the results of the two calculations is relatively small for values of b/a between 0.7 and 1.0, and that the differences do not begin to increase until b/a is less than 0.7. In view of the factors mentioned during the calculation for the infinitely long plate, this increase in the difference as b/a decreases is quite understandable. For the square plate, the coefficient A_{11} is the largest of all coefficients. On the other hand, as the plate gets longer, the coefficients A_{mn} with larger m begin to increase, so that the coefficients with indices m = 2 or = 3 or = 4 (and with even larger m for correspondingly long plates) become the largest of all the coefficients A_{mn} and thus can no longer be neglected, as was the case when only indices 1 through 3 were included. If one wishes to take only three indices in order to simplify the calculation, it is better in the case of a long plate to take not indices 1 through 3 but instead, depending on the ratio between the sides, e.g. the indices 2 through 4 or even larger numbers for the longitudinal direction. Fig. 6 depicts the difference in the results of two calculations with indices $m = 1$ through 3 and $m = 2$ through 4 ($n = 1$ through 3 in both cases). Up to roughly $b/a = 0.37$, the calculation with indices 2 through 4 yields smaller, i.e. better approximations for the critical shear load. For $b/a \approx 0.37$, both curves yield the same value c_a which seems somewhat large in comparison with the value c_a for adjacent values of b_a. Thus if $b/a \approx 0.37$, taking only three indices in the longitudinal direction probably supplies relatively poorer results than for values of b/a in /184 adjacent regions. If the curves are calculated for indices 1 through 4 in the longitudinal direction and 1 through 3 in the transverse direction, the resulting values for c_a are smaller than those from either of the two previously calculated curves (although some of the differences are vanishingly small) for all values of b/a. As might have been expected, the maximum difference occurs near $b/a = 0.37$, the intersection of the two previous curves. However, even for the value b/a at which the curve calculated with indices $m = 2$ through 4 reaches it minimum c_a, the difference is not insignificantly small. Accordingly, as was also indicated by the results of the various approximations made in the case of the infinitely long plate, the best possible approximations would be obtained by following a previous suggestion of Reissner for decreasing b/a (i.e. for longer plates) and increasing only the number of indices m in the longitudinal direction.

Nevertheless, the calculation was not executed in this manner; since first the equations for calculating the curves c_a (b/a) become quite complicated with four indices m, even if there are only three indices n. Hence, calculating these curves involves a great deal more computation than was necessary

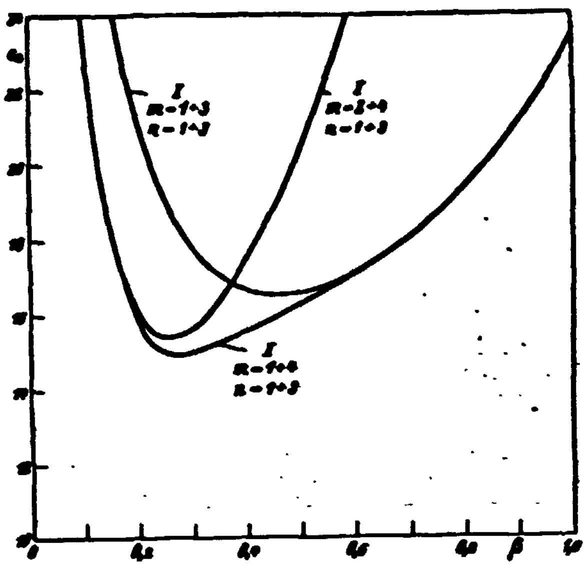

Fig. 6. $c_a = t_{cr}(b/2)^2/D$ (Coefficient of the critical shear load of an isotropic plate) as a function of the ratio $\beta = b/a$ between the sides. Comparison of different approximate solutions.

in the case of three indices in two directions[3]. Moreover, the difference in the results obtained with five indices as compared with those obtained with three indices is much larger for the infinitely long plate than for the square plate; for ratios b/a lying /185 between 0 and 1, it might therefore be assumed that the difference will be smaller than for the infinitely long plate.

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[3]Just how great the difference in computation is can be seen e.g. from the equations for the three curves depicted in Fig. 6. These equations read

for m = 1 through 3 and n = 1 through 3:

$$c_a = \frac{\pi^4}{128}\,\frac{225\left(\beta + \frac{1}{\beta}\right)}{\sqrt{\frac{706}{(\beta^2+1)^2} + 2025\left[\frac{1}{(9\beta^2+1)^2} + \frac{1}{(\beta^2+9)^2}\right]}};$$

for m = 2 through 4 and n = 1 through 3:

$$c_a = \frac{\pi^4}{128}\,\frac{1}{\beta\sqrt{\left\{\left[\frac{1}{5(\beta^2+1)}\right]^2 + \left[\frac{1}{(9\beta^2+1)}\right]^2\right\}\left\{\left[\frac{1}{5(\beta^2+1)}\right]^2 + \left[\frac{2}{7(4\beta^2+1)}\right]^2\right\}}}.$$

For numerical calculations, these equations are relatively simple when compared with the equation:

for m = 1 through 4 and n = 1 through 3:

$$c_a = \frac{\pi^4}{128}\,\frac{1}{\beta\sqrt{s_1 + \sqrt{s_1^2 - s_0}}};$$

where:

$$s_1 = \frac{1}{2(\beta^2+1)^2}\,s' + \frac{2}{(4\beta^2+1)^2}\,s'',$$

$$s_0 = \frac{4}{(\beta^2+1)^2(4\beta^2+1)^2}\left\{s's'' - \left[\left(\frac{1}{81} - \frac{1}{175}\right)\frac{1}{(\beta^2+1)^2} + \frac{1}{25}\,\frac{1}{(\beta^2+9)^2} - \frac{1}{7}\,\frac{1}{(9\beta^2+1)^2}\right]^2\right\}.$$

TABLE 7a and b. SYSTEMS OF COEFFICIENTS FOR THE EQUATIONS (8)

a) q odd: Case I
q even: Case II

$q+m_q, n$ \ $A_{(q+m_q),n}$	$A_{q,1}$	$A_{q,3}$	$A_{q+2,1}$	$A_{q+2,3}$	$C\cdot A_{q+1,2}$
$q, 1$	$\frac{\varphi(q,1)}{q\cdot 1\cdot\beta}$	0	0	0	$\frac{q+1}{-2q-1}\cdot\frac{2}{-3}$
$q, 3$	0	$\frac{\varphi(q,3)}{q\cdot 3\cdot\beta}$	0	0	$\frac{q+1}{-2q-1}\cdot\frac{2}{+5}$
$q+2, 1$	0	0	$\frac{\varphi(q+2,1)}{(q+2)\cdot 1\cdot\beta}$	0	$\frac{q+1}{+2q+3}\cdot\frac{2}{-3}$
$q+2, 3$	0	0	0	$\frac{\varphi(q+2,3)}{(q+2)\cdot 3\cdot\beta}$	$\frac{q+1}{+2q+3}\cdot\frac{2}{+5}$
$q+1, 2$	$\frac{q}{+2q+1}\cdot\frac{1}{+3}$	$\frac{q}{+2q+1}\cdot\frac{3}{-5}$	$\frac{q+2}{-2q-3}\cdot\frac{1}{+3}$	$\frac{q+2}{-2q-3}\cdot\frac{3}{-5}$	$\frac{\varphi(q+1,2)}{(q+1)\cdot 2\cdot\beta}C^{-2}$

b) q even: Case I
q odd: Case II

$q+m_q, n$ \ $A_{(q+m_q),n}$	$A_{q+1,1}$	$A_{q+1,3}$	$C\cdot A_{q,2}$	$C\cdot A_{q+2,2}$
$q+1, 1$	$\frac{\varphi(q+1,1)}{(q+1)\cdot 1\cdot\beta}$	0	$\frac{q}{+2q+1}\cdot\frac{2}{-3}$	$\frac{q+2}{-2q-3}\cdot\frac{2}{-3}$
$q+1, 3$	0	$\frac{\varphi(q+1,3)}{(q+1)\cdot 3\cdot\beta}$	$\frac{q}{+2q+1}\cdot\frac{2}{+5}$	$\frac{q+2}{-2q-3}\cdot\frac{2}{+5}$
$q, 2$	$\frac{q+1}{-2q-1}\cdot\frac{1}{+3}$	$\frac{q+1}{-2q-1}\cdot\frac{3}{-5}$	$\frac{\varphi(q,2)}{q\cdot 2\cdot\beta}C^{-2}$	0.
$q+2, 2$	$\frac{q+1}{+2q+3}\cdot\frac{1}{+3}$	$\frac{q+1}{+2q+3}\cdot\frac{3}{-5}$	0	$\frac{\varphi(q+2,2)}{(q+2)\cdot 2\cdot\beta}C^{-2}$

where z' and z" are to be calculated from

$$z' = \left(\frac{1}{81}+\frac{1}{625}\right)\frac{1}{(\beta^2+1)^2}+\frac{1}{25}\left[\frac{1}{(\beta^2+9)^2}+\frac{1}{(9\beta^2+1)^2}\right].$$

$$z'' = \frac{1}{25}\left(\frac{1}{81}+\frac{1}{625}\right)\frac{1}{(\beta^2+1)^2}+\frac{1}{625}\,\frac{1}{(\beta^2+9)^2}+\frac{1}{49}\,\frac{1}{(9\beta^2+1)^2}.$$

Furthermore, in view of the practical significance of the entire problem, which will be discussed in more detail later [9], it doesn't make much sense to determine the curve c_a (b/a) with extreme precision. Therefore, it appears perfectly sufficient to calculate the various approximations c_a (b/a) for three successive indices m in the longitudinal direction and for the indices n = 1 through 3 in the transverse direction, and to estimate the desired exact curves c_a (b/a) on the basis of these approximations. The difference between the approximate curves calculated with three indices and the exact values will be greatest in the extreme case b/a = 0, i.e. for the infinitely long plate, in which the difference is less than 12%, calculated earlier. Over its entire range, the errors in the estimated curve should amount to only a fraction of this difference. If /126 the indices in the longitudinal direction are designated m = q, (q + 1) and (q + 2) and those in the transverse direction n = 1 through 3, (8) can be used to acquire the systems of equations given in Table 7a and 7b. These systems of equations are solved in the previous manner, resulting in the following two equations for c_a, using the symbols given in (7):

$$c_a = \frac{\pi^4}{128}\frac{\sqrt{\varphi(q+1,2)}}{2(q+1)\beta}\frac{1}{\sqrt{\left(\frac{q}{2q+1}\right)^2\left[\frac{1}{9\varphi(q,1)}+\frac{9}{25\varphi(q,3)}\right]+\left(\frac{q+2}{2q+3}\right)^2\left[\frac{1}{9\varphi(q+2,1)}+\frac{9}{25\varphi(q+2,3)}\right]}} \quad (14)$$

(Case I, when q is odd),(Case II, when q is even)

and

$$c_a = \frac{\pi^4}{128}\frac{1}{2(q+1)\beta}\frac{1}{\sqrt{\left[\frac{1}{9\varphi(q+1,1)}+\frac{9}{25\varphi(q+1,3)}\right]\left[\frac{q^2}{(2q+1)^2\varphi(q,2)}+\frac{(q+2)^2}{(2q+3)^2\varphi(q+2,2)}\right]}}$$

(Case I, when q is even), (Case II, when q is odd) (15)

Here, in accordance with (7)

$$\varphi(q+m_1,n) = [(q+m_1)^2\beta^2+n^2]^2.$$

The approximate curves in Fig. 7 were calculated with the aid of Eq. (14) and (15).

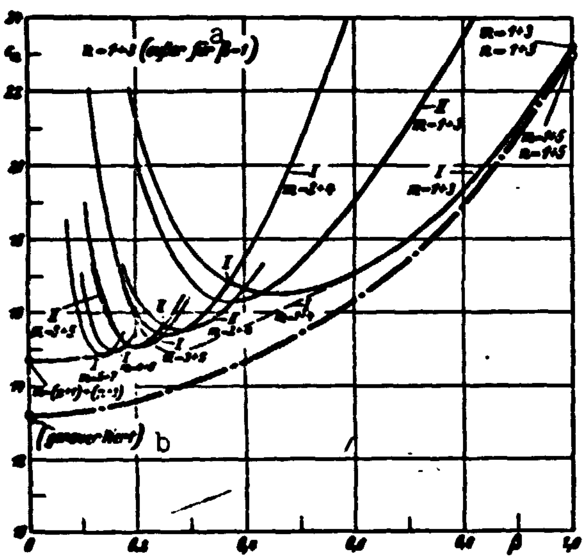

Fig. 7. Coefficient $c_a = t_{cr}(b/2)^2/D$ of critical shear load of isotropic rectangular plate as a function of the ratio $\beta = b/a$ between the sides. Using the curves calculated for various approximate solutions and the known exact solution for $\beta = 0$, the dash-dot line (-·-·-·) gives the conjectured path of an exact curve.
Key: a. Except for b. Exact value

b) Fundamental criteria for wave form (Cases I and II).

Cases I and II differ in that only unknowns A_{mn} with even sum $(m + n)$ occur in equation system I and only those with odd sum $(m + n)$ occur in system II. In the result, the new cases differ in that buckling I is a maximum (w_{max}) in the center of plate, while there is a nodal line ($w = 0$) through the center of the plate in Case II. The approximations now show (at first apparently quite irregularly) that sometimes Case I and other times Case II yields the smaller critical shear load, depending on the ratio b/a. This is a circumstance that could apply to both the approximation and the exact solution,as will be suggested by the following analysis.

Suppose we wished to calculate the critical shear load of an infinitely long plate of constant width b, supported freely along the longitudinal edges, and subject to the further /187
condition that certain points A along the center line of the plate spaced at regular intervals a' should not experience any deflection (cf. Fig. 8). Solutions corresponding to this problem are contained in the curves of Fig. 2 ("exact solution"),

representing the Southwell-Skan solution for the infinitely long plate. If $a' \approx 1.25b$ (a' as in Fig. 8), then $c_a = 13.165$ ($= c_{a, min}$). In this case, the half-wavelength is $\ell = a'$. If a' increases, c_a also increases in accordance with the curve $c_a(\ell/b)$, since initially $\ell = a'$ still holds. At $a' \approx 1.8b$, $c_a = 14.2$. The same value c_a is also obtained for $\ell = 0.9b$; i.e. in this case, the same value c_a is obtained for both $\ell = a'$ and for $\ell = a'/2$. Therefore, two buckling shapes are possible under the same shear load. If a' goes beyond the value 1.8b, the resulting buckling shape has two half waves in the interval a', i.e. $\ell = a'/2$, and this shape always gives the smallest value for c_a. Initially, c_a decreases from 14.2 to 13.165 for $a' \approx 2.5b$ (i.e. $\ell = 1.25b$), and then c_a increases with increasing $\underline{a}$ back up to $c_a = 13.6$. Namely, for $a' \approx 3.0b$, this value of c_a corresponds to both the buckling form with two half-waves ($\ell \approx 1.5b$) and the buckling form with 3 half-waves ($\ell \approx 1.0b$). According to this analysis, therefore, the (whole) number m of half-waves in the interval a' increases with increasing a'; for each number $m = a'/\ell$, c_a can assume values between $c_{a, min} = 13.165$ and a larger value of c_a which decreases as m increases. The largest such c_a obtained for m = 1 (or m = 2) with $c_a = 14.2$. For odd numbers m, the buckling is greatest (w_{max}, Case I) at the point M lying halfway between two points A. For even m, there is a nodal line (w = 0, Case II). With increasing a', Case I and Case II alternately give the smallest shear load (c_a).

Passing from the specific problem to the general problem under investigation, we acquire an additional condition, namely that the buckling not only at the points A, but also on the straight lines passing through the points A and perpendicular to the longitudinal edges must be equal to 0 and that the plate is supported without stress along these lines. This supplementary condition naturally will cause substantial changes in the /188

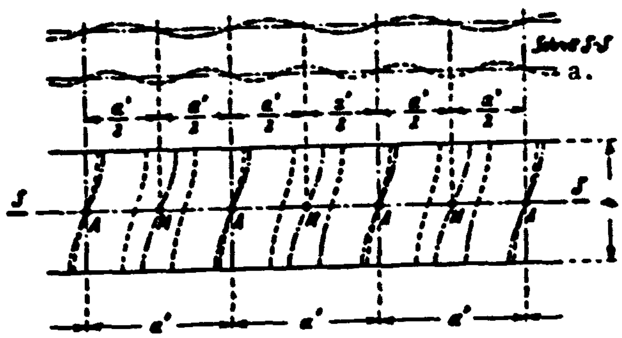

Fig. 8. Various possible buckling shapes of an infinitely long plate. If the deflection is to be w = 0 at points A, the deflection at points M (halfway between two points A) can be a maximum w_{max}, or else a nodal line (w = 0) may pass through M.

Key: a. Section

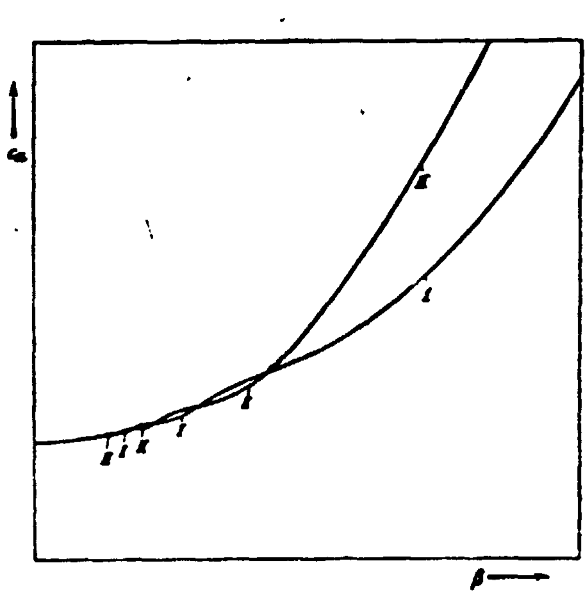

Fig. 9. Hypothetical curves for exact c_a as a function of β. Curve I corresponds to the buckling form with greatest deflection w_{max} in the center of the plate, while Curve II corresponds to a nodal line (w = 0) passing through the point in the center of the plate.

buckled shape and thus in the magnitude of the associated shear load. The smaller the ratio a/b, the greater the changes. Nevertheless, it is still altogether possible that, as in the problem just discussed (cf. Fig. 8), which is somewhat simpler than our present problem, Cases I and II will provide the smaller (critical) shear load in alternation with decreasing ratio b/a even for the plate freely supported along all four edges. This would consistent with the results of the previous approximation. The curves for the critical shear load as a function of the ratio of the sides -- $c_a(b/a)$ -- valid for Cases I and II might then look like those shown in Fig. 9. The only parts of the curves which are of actual practical importance are those corresponding to the smaller of the two values of c_a for a given side ratio. The estimated curve depicted in Fig. 7 should be derived only from these practically important portions of the curves. This estimated curve should provide values of c_a accurate roughly to three places, so that the accuracy ought to be sufficient in practice.

6. Solution for the orthotropic plate. For an orthotropic plate, the critical shear load is calculated by one of the two formulas corresponding to Eqs. (4a) and (4b):

$$t_{cr} = c_a \ (D_1 D_2^3)^{1/4} \ (b/2)^{-2} \qquad (16a)$$

or

$$t_{cr} = c_b \ (D_3 D_2)^{1/2} \ (b/2)^{-2} \qquad (16b)$$

(For the special case of the isotropic plate, Eq. (16a) becomes (13)).

The coefficients c_a and c_b depend on the parameter θ of the orthotropic plate as given by (3). To determine the critical shear load of an orthotropic plate, the best way is to first calculate the parameter θ and then find the associated coefficient c_a or c_b. These coefficients are obtained by the same method used for calculating the coefficient c_a of the isotropic plate in the preceding sections; however, the starting point is Eqs. (4a) and (5a) or (4b) and (5b) instead of Eqs. (7) and (8). Except for the fact that c_a and c_b are now functions of β_a and β_b -- cf. (4a) or (4b) -- and not just of $\beta = b/a$, the only major difference relative to the isotropic plate is in the terms ϕ_a (m, n) and ϕ_b (m, n), which occur in Eqs. (5a) and (5b) in place of the term ϕ (m, n) in (8). All of the equations for c_a for the isotropic plate derived from Eq. (8) also hold for the coefficients c_a and c_b of the orthotropic plate, as long as no specific expression for ϕ (m, n) -- and thus no specific value for β -- is substituted in these equations; ϕ and β are replaced by ϕ_a or ϕ_b and β_a or β_b. In particular, Eqs. (14) and (15) apply to the isotropic plate, as does Eq. (9) if, in the latter equation, $\phi(\alpha_n)$ is replaced by the expression

$$\varphi_a(\alpha, \varkappa) = \alpha^4 + 2\frac{\alpha^2 \varkappa^2}{\theta} + \varkappa^4 \tag{17a}$$

or the expression

$$\varphi_b(\alpha, \varkappa) = \theta^2 \alpha^4 + 2\alpha^2 \varkappa^2 + \varkappa^4 \tag{17b}$$

The coefficients c_a calculated for the isotropic plate applied to an orthotropic plate is the parameter $\theta = 1$ (i.e. $c_a = c_b$). Apart from this case, the coefficient c_a has been calculated /189
from (14) and (15) only for the parameters $\theta = 2$ and $\theta = \infty$ (replacing ϕ by ϕ_a and β by β_a). In general, these calculations will be sufficient, together with exact values calculated in a previous work [6] for an infinitely long plate and for any arbitrary value of $\theta \geq 1$ and for arbitrary b/a, to determine the coefficient c_a of the critical shear load (16a) with adequate precision. For parameters $\theta < 1$ or in case a more precise result is desired for a parameter $\theta > 1$, the calculation can be carried out just as for $\theta = 2$ and for $\theta = \infty$.

The depiction of the curve c_a (β_a) can be restricted to the region $0 \leq \beta_a \leq 1$. If, in a specific case, a value $\beta_a > 1$ is obtained, the designations of the sides and of the rigidities D_1 and D_2 are interchanged. It should be observed that in certain cases (e.g. when D_2 is considerably larger than D_1 and $\beta_a = 1$ or β_a is only slightly less than 1), b will be the longer side of the rectangular plate in (16a). In the limiting case $\theta = \infty$, as in the extreme case $\theta = 0$ for infinitely long plate (b/a $\to$ 0), we use the equations corresponding to Eq. (12).

$$c_a = 0{,}1614920 \frac{\frac{1}{\alpha}\sqrt{\alpha^4 + 16}}{\sqrt{\frac{1}{81}\frac{1}{\alpha^4 + 1} + \frac{1}{25}\frac{1}{\alpha^4 + 81}}} \qquad \text{(for } \theta = \infty\text{)} \qquad (18a)$$

$$c_b = 0{,}1614920 \frac{\frac{2}{\alpha}\sqrt{2\alpha^2 + 4}}{\sqrt{\frac{1}{81}\frac{1}{2\alpha^2 + 1} + \frac{1}{25}\frac{1}{18\alpha^2 + 81}}} \qquad \text{(for } \theta = 0\text{)} \qquad (18b)$$

and find for $\alpha = 1.0$ (instead of the exact value 0.975) or for $\alpha = 1.04$, values c_a and c_b only 0.4% greater than the values $c_a = 8.125$ or $c_b = 11.708$ corresponding to the exact solution [4], representing the minima of the exact curve c_a or c_b as function of half-wavelengths of the buckled surface. The corresponding approximation calculated by (12) for $\theta = 1$ also has an error of only about 0.4% (cf. Table 2). Hence, the approximation appears to furnish equally good results for all values of θ in the limiting case of the infinitely long plate.

On the other hand, the result in the other extreme case, namely $\beta_a = 1$, does not appear to be as good for $\theta > 1$ as for $\theta = 1$. While the value of c_a dropped from 23.25 to 23.05, i.e. by only 0.9% (cf. Table 5) for the square isotropic plate ($\beta = 1$; $\theta = 1$) when five indices m and n were taken instead of three, the correspondingly calculated value of c_a for $\theta = \infty$ and $\beta_a = 1$ dropped from 12.07 to 11.735, i.e. by 2.85%. For $\beta_a = 1$, the convergence appears to be better for $\theta = 1$ than for $\theta > 1$; however, for $\theta = \infty$ and $\beta_a = 1$, the calculated approximation $c_a = 11.735$ still appears sufficiently accurate. In addition, as mentioned previously, approximate values c_a have been calculated vs. β_a for $\theta = 2$ and $\theta = \infty$, taking three indices each in the

[4] [6], Table I. According to Column 9 of this Table, the values α were calculated as follows:

$$\text{for } \theta = \infty:\ \alpha = 2/2.0515 = 0.975 \approx 1.0$$

$$\theta = 0:\ \alpha = 2/1.9212 = 1.04$$

longitudinal and transverse directions. The calculated curves /190 are depicted in Fig. 10. Based on these approximations and exact values available for the limiting case $\beta_a \to 0$ [6], the estimated curves depicted in Fig. 10 for c_a (β_a) were constructed. Since the curve for $\theta = 2$ is almost exactly halfway between the curves for $\theta = 1$ and for $\theta = \infty$ with great regularity for all values of β_a, it can be assumed that the curves c_a (β_a) for other values θ than $\theta = 2$ will run with similar regularity between the curves for $\theta = 1$ and for $\theta = \infty$. For any arbitrary value θ, the exact value c_a can be given for $\beta_a = 0$ [5]. Starting from this point, the approximate curve c_a (β_a) for any arbitrary value θ can be drawn into Fig. 10.

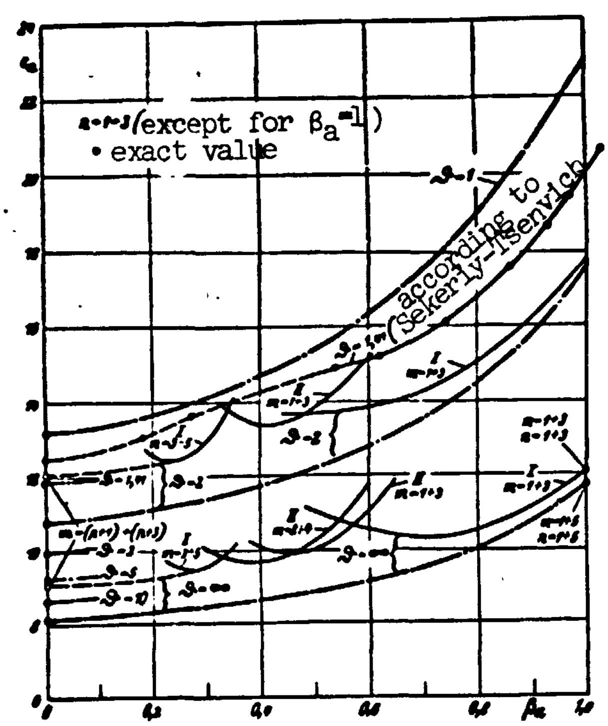

Fig. 10. Coefficient $c_a = t_{cr}(b/2)^2/(D_1D_2^3)^{1/4}$ of the critical shear load of an orthogonal-anisotropic plate as a function of $\beta_a = (b/a)(D_1/D_2)^{1/4}$. The precise curves have been estimated for $\theta = 2$ and for $\theta = \infty$ in the same way as in Fig. 7 for $\theta = 1$.

This diagram also includes a number of points for $\theta = 1.41$, obtained from the calculation of Sekeriy-Tsenkovich [7], which in general corresponds to the approximation with three indices each in the longitudinal and transverse direction. For a plywood board of thickness h, Sekeriy-Tsenkovich chose moduli of elasticity for bending deformation $E_1 = 1.4 \cdot 10^2$ kg/cm^2 in one direction and $E_2 = E_1/12$ in the other, $G = 0.12 \cdot 10^5$ kg/cm^2 for the shear

[5] Cf. [6], Fig. 2a. In Fig. 10 of this work, the precise value c_a for $\theta = 1.41$, 3, 5 and 10 are entered along with those for $\theta = 1$, 2 and ∞.

modulus (of torsion deformation), and $\nu_x = 0.46$ and $\nu_y = \nu_x/12$ for the Poisson ratios, and thus obtained by (1a):

$$D_1 = 12{,}000\ h^3\ \text{kg/cm}^2;\ D_2 = 1000\ h^3\ \text{kg/cm}^2,\ \text{and}$$

$$D_3 = 2456\ h^3\ \text{kg/cm}^2\ \text{and according to (3):}\ \theta = 1.41.$$

The approximate value of c_a obtained by the Sekeriy-Tsenkovich calculation for $\theta = 1.41$ is quite consistent with the approximate curves found for $\theta = 1$, 2 and ∞.

7. Summary. The critical shear load t_{cr} for an isotropic rectangular plate supported without stress at the edg's can be calculated by (13). In this equation, b is the width of the plate, D is the plate rigidity obtained from (6), and c_a is a coefficient taken from the diagram in Fig. 7. Analogously, /191 (16a) applies to the case of an orthogonal-anisotropic plate; the coefficient c_a occurring in (16a) is depicted in the diagram of Fig. 10. The rigidities D_1, D_2 and D_3 of the orthogonal-anisotropic plate are given by Eq. (1a). The determination of the coefficients c_a a d c_b is described in Section 6. When the parameter θ given by (3) is greater than unity, it is best to use Eq. (16a); on the other hand, Eq. (16b) is better for parameters $\theta < 1$.

A detailed study has shown that the coefficient c_a of the critical shear load calculated in this work on the basis of an approximation theory should be sufficiently accurate (the error probably does not occur until the second place following the decimal point), particularly since theoretical assumptions are almost never satisfied exactly; for instance, as the result of completed experiments have indicated, a very slight initial bulge (practically unavoidable) has an especially great disturbing effect in the case of very thin plates.

These experiments as well as numerical examples of the problem discussed in this work will be communicated in a further report [9].

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